Subset-Saturated Cost Partitioning for Optimal Classical Planning: Additional Details

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This brief paper contains some additional proof details referenced from Seipp and Helmert (2019).

We remind the reader that we use $\sum$ and $+$ to denote left summation and $\bigoplus$ and $\oplus$ to denote path summation (Seipp and Helmert, 2019).

1 Abstraction Heuristics with General Cost Functions are Admissible and Consistent

We state in Section "Heuristics" that all abstraction heuristics are admissible and consistent even if we allow negative and infinite costs. We now show this result.

First, we recall the definition of goal distances for a transition system $\mathcal{T}$, cost function $\mathit{cost} \in \mathcal{C}(\mathcal{T})$ and state $s \in S(\mathcal{T})$:

$$h^*_{\mathcal{T}}(\mathit{cost}, s) = \inf_{\pi \in \Pi_\star(\mathcal{T}, s)} \mathit{cost}(\pi),$$

where $\Pi_\star(\mathcal{T}, s)$ is the set of goal paths from s in $\mathcal{T}$ and $\inf \emptyset$ is defined as ∞. We write h^* for $h^*_{\mathcal{T}}$ where $\mathcal{T}$ is clear from context or does not matter.

We say that a heuristic h is *goal-aware* if $h(\mathit{cost}, s) \leq 0$ for all cost functions *cost* and goal states s.

We first show that all heuristics that are goal-aware and consistent are admissible, and then we conclude the proof by showing that abstraction heuristics (with general cost functions) are goal-aware and consistent.

1.1 Goal-Aware + Consistent $\Longrightarrow$ Admissible

It is well-known that in the setting of finite non-negative cost functions, a heuristic that is both goal-aware and consistent is admissible. We now show that this result also holds for general cost functions.

Let $\mathcal{T}$ be a transition system, and let h be a heuristic for $\mathcal{T}$ that is goal-aware and consistent. We show that h is admissible.

Let $s \in S(\mathcal{T})$ and $\mathit{cost} \in \mathcal{C}(\mathcal{T})$. We must show $h(s) \leq h^*(s)$. Because $h^*(s)$ is defined as the infimum of the costs of all goal paths for s, it is sufficient to show that $h(s) \leq \mathit{cost}(\pi)$ for all goal paths π for s.

Let $\pi = \langle s \xrightarrow{\ell_1} s^1, \dots, s^{n-1} \xrightarrow{\ell_n} s^n \rangle$ be such a goal path from s to a goal state s_n. Because h is consistent we have $h(s) \leq \mathit{cost}(\ell_1) \oplus h(s_1) \leq \mathit{cost}(\ell_1) \oplus \mathit{cost}(\ell_2) \oplus h(s_2) \leq \dots \leq \mathit{cost}(\ell_1) \oplus \dots \mathit{cost}(\ell_n) \oplus h(s_n) = \mathit{cost}(\pi) \oplus h(s_n)$. Since h is goal-aware, we have $h(s_n) \leq 0$ and therefore $h(s) \leq \mathit{cost}(\pi)$.

1.2 Abstraction Heuristics with General Cost Functions are Goal-Aware

Let h be an abstraction heuristic for transition system $\mathcal{T}$ with abstraction mapping α and abstract transition system $\mathcal{T}'$. Let $\mathit{cost} \in \mathcal{C}(\mathcal{T})$ and $s \in S_\star(\mathcal{T})$. We must show $h(\mathit{cost}, s) \leq 0$.

By definition of abstraction heuristics, we have $h(\mathit{cost}, s) = h^*_{\mathcal{T}'}(\mathit{cost}, \alpha(s))$. Because s is a goal state of $\mathcal{T}$, $\alpha(s)$ is a goal state of $\mathcal{T}'$. (This is one of the properties required of abstraction mappings; cf. Helmert et al., 2007.) Because the empty path is a goal path for $\alpha(s)$ with cost 0, we have $h^*_{\mathcal{T}'}(\mathit{cost}, \alpha(s)) \leq 0$, showing that h is goal-aware.

1.3 Abstraction Heuristics with General Cost Functions are Consistent

Let h be an abstraction heuristic for transition system $\mathcal{T}$ with abstract transition system $\mathcal{T}'$. Let $\mathit{cost} \in \mathcal{C}(\mathcal{T})$ and $s \xrightarrow{\ell} s' \in T(\mathcal{T})$. We must show $h(\mathit{cost}, s) \leq \mathit{cost}(\ell) \oplus h(\mathit{cost}, s')$.

Because abstractions preserve transitions (Helmert et al., 2007), we have $\alpha(s) \xrightarrow{\ell} \alpha(s') \in T(\mathcal{T}')$. We thus obtain $h(\mathit{cost}, s) = h^*_{\mathcal{T}'}(\mathit{cost}, \alpha(s)) = \inf_{\pi \in \Pi_\star(\mathcal{T}', \alpha(s))} \mathit{cost}(\pi) \leq \inf_{\pi' \in \Pi_\star(\mathcal{T}', \alpha(s'))}(\mathit{cost}(\ell) \oplus \mathit{cost}(\pi')) = \mathit{cost}(\ell) \oplus \inf_{\pi' \in \Pi_\star(\mathcal{T}', \alpha(s'))} \mathit{cost}(\pi') = \mathit{cost}(\ell) \oplus h(s')$, where the inequality holds because for every goal path $\pi' \in \Pi_\star(\mathcal{T}', \alpha(s'))$ the path π that consists of ℓ followed by π' is a goal path for $\alpha(s)$. This proves the result.

2 Cost Partitioning with General Cost Functions: Consistency

Let $h_1, \dots, h_n$ be consistent heuristics for transition system $\mathcal{T}$, let $\mathit{cost} \in \mathcal{C}(\mathcal{T})$, and let $\mathit{cost}_1, \dots, \mathit{cost}_n \in \mathcal{C}(\mathcal{T})$ such that $\sum_{i=1}^n \mathit{cost}_i(\ell) \leq \mathit{cost}(\ell)$ for all $\ell \in L(\mathcal{T})$. We must show that the cost-partitioned heuristic h for $\mathcal{T}$ is consistent under cost function cost, i.e., $h(\mathit{cost}, s) \leq \mathit{cost}(\ell) \oplus h(\mathit{cost}, s')$ for all $s \xrightarrow{\ell} s' \in T(\mathcal{T})$, where $h(\mathit{cost}, s) := \sum_{i=1}^n h_i(\mathit{cost}_i, s)$. We emphasize that we place no restrictions on the cost functions $\mathit{cost}, \mathit{cost}_1, \dots, \mathit{cost}_n$, i.e., negative and (positively or negatively) infinite costs are permitted.

We obtain

$$\begin{aligned}
h(\mathit{cost}, s) &= \sum_{i=1}^n h_i(\mathit{cost}_i, s) && \text{(definition of } h\text{)} \\
&\leq \sum_{i=1}^n (\mathit{cost}_i(\ell) \oplus h_i(\mathit{cost}_i, s')) && \text{(consistency of } h_i\text{)} \\
&\leq \sum_{i=1}^n \mathit{cost}_i(\ell) \oplus \sum_{i=1}^n h_i(\mathit{cost}_i, s') && \text{(main step of the proof, see below)} \\
&\leq \mathit{cost}(\ell) \oplus \sum_{i=1}^n h_i(\mathit{cost}_i, s') && \text{(cost partitioning condition)} \\
&= \mathit{cost}(\ell) \oplus h(s') && \text{(definition of } h\text{),}
\end{aligned}$$

which proves the result.

It remains to show the main step of the proof: we must demonstrate LHS $\leq$ RHS, where

$$\text{LHS} = \sum_{i=1}^n (\mathit{cost}_i(\ell) \oplus h_i(\mathit{cost}_i, s')) \quad \text{and}$$

$$\text{RHS} = \sum_{i=1}^n \mathit{cost}_i(\ell) \oplus \sum_{i=1}^n h_i(\mathit{cost}_i, s').$$

If $cost_i(\ell)$ and $h_i(cost_i, s')$ are finite for all $1 \leq i \leq n$, then all terms in the definition of LHS and RHS are finite, in which case it is easy to see LHS = RHS. Otherwise, let $i_0 \in \{1, \dots, n\}$ be the smallest index for which at least one of $cost_{i_0}(\ell)$ and $h_{i_0}(cost_{i_0}, s')$ is (positively or negatively) infinite.

If $cost_{i_0}(\ell) = \infty$, we get $\sum_{i=1}^{n} cost_i(\ell) = \infty$ and therefore RHS $= \infty$. Similarly, if $h_{i_0}(cost_{i_0}, s') = \infty$, we get $\sum_{i=1}^{n} h_i(cost_i, s') = \infty$ and again RHS $= \infty$. In both cases LHS $\leq$ RHS holds trivially because $x \leq \infty$ for all x.

So it remains to consider the case where neither $cost_{i_0}(\ell)$ nor $h_{i_0}(cost_{i_0}, s')$ equals ∞, but at least one of them is infinite. Then this must be a negative infinity, and we obtain $cost_{i_0}(\ell) \oplus h_{i_0}(cost_{i_0}, s') = -\infty$. Because i_0 is the first index for which we get infinities, we also know that $cost_j(\ell) \oplus h_j(cost_j, s')$ is finite for all $j < i_0$. Together, these two facts show LHS $= -\infty$, from which LHS $\leq$ RHS follows trivially because $-\infty \leq x$ for all x. This concludes the proof.

3 Cost Partitioning with General Cost Functions: Admissibility

Let $h_1, \dots, h_n$ be admissible heuristics for transition system $\mathcal{T}$, let $cost \in \mathcal{C}(\mathcal{T})$, and let $cost_1, \dots, cost_n \in \mathcal{C}(\mathcal{T})$ such that $\sum_{i=1}^{n} cost_i(\ell) \leq cost(\ell)$ for all $\ell \in L(\mathcal{T})$. We must show that the cost-partitioned heuristic h for $\mathcal{T}$ is admissible under cost function *cost*, i.e., $h(cost, s) \leq h^*(cost, s)$ for all $s \in S(\mathcal{T})$, where $h(cost, s) := \sum_{i=1}^{n} h_i(cost_i, s)$. Again, no restrictions are placed on the cost functions.

We define

$$h'(cost, s) := \sum_{i=1}^{n} h^*(cost_i, s)$$

and view h' as a cost-partitioned heuristic under cost function *cost* whose component heuristics are all h^*. The heuristic h' is clearly goal-aware, and from Section 2, it is consistent (because h^* is consistent). As shown in Section 1.1, h' is therefore admissible.

We get:

$$\begin{aligned}
h(cost, s) &= \sum_{i=1}^{n} h_i(cost_i, s) && \text{(definition of } h\text{)} \\
&\leq \sum_{i=1}^{n} h^*(cost_i, s) && \text{(because all } h_i \text{ are admissible)} \\
&= h'(cost, s) && \text{(definition of } h'\text{)} \\
&\leq h^*(cost, s) && \text{(because } h' \text{ is admissible),}
\end{aligned}$$

proving that h is admissible.

References

Malte Helmert, Patrik Haslum, and Jörg Hoffmann. Flexible abstraction heuristics for optimal sequential planning. In *Proc. ICAPS 2007*, pages 176–183, 2007.

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